

DRNLA: Dual Rewriting for Branching-Time Verification of Non-Linear Arithmetic Programs

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Solving the Problem

Goal

- Want to verify temporal properties of programs with **polynomials**?
- Unfortunately existing CTL tools **are unsound!** (T2, FUNCTION)

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Our Results

- We can make these tools support nonlinear arithmetic (NLA).
- We will show you a method to synthesize linear integer arithmetic (LIA) replacements for NLA expressions.
- *Dual Rewriting* algorithm, with dynamic and static analysis.
- New tool DRNLA.
- After pre-processing with DRNLA, existing CTL verifiers can be employed.

A program with a polynomial loop guard

```
1 int y=1, z=6, c=0, p=2;
2 int k=*;
3 while (z*z - 12y - 6z + 12 + c <= k) :
4     y = y + z;
5     z = z + 6;
6     c = c + 1;
7     p = 1;
8 p = 0;
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What properties can we prove of such programs?

- Reachability, e.g. DIG (TOSEM'14).
- Termination, e.g. DynamiTe (OOPSLA'20) and ν Term (FSE'22).
- LTL, some support, e.g. Ultimate.
- CTL, none.

A program with a polynomial loop guard

Our work: Synthesize LIA replacements for NLA expressions!

```
int y=1, z=6, c=0, p=2;
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while(z*z - 12y - 6z + 12 + c <= k) :
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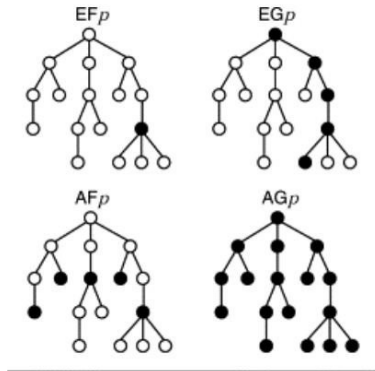
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    c = c + 1;
    p = 1;
p = 0;
return 0;
```

```
int y=1, z=6, c=0,
    p=2;
int k=*;
while(c <= k) :
    y = y + z;
    z = z + 6;
    c = c + 1;
    p = 1;
p = 0;
return 0;
```

Outcome: Can preprocess NLA programs and then use CTL verifiers.

What is special about CTL? A Reminder.

Safety and liveness properties, with particular focus on **branching**.



(LTL has no branching; implicitly over all paths, one at a time.)

CTL Example

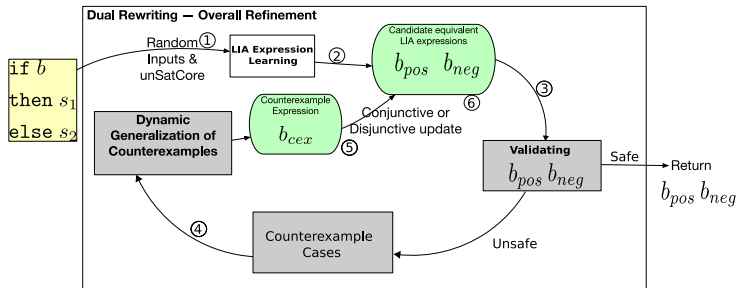
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Valid: $EF(p = 0) \wedge EF(p = 1)$

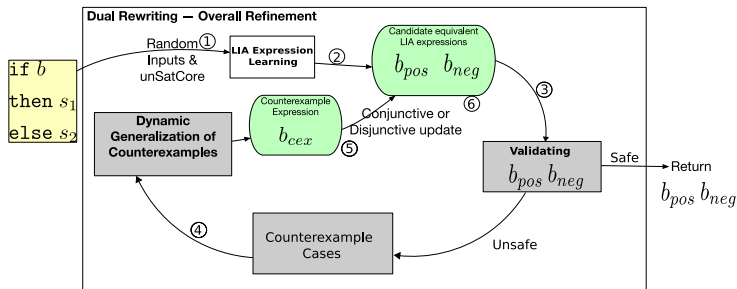
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3 while ( z*z-12y-6z+12+c<=k ):
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5     z = z + 6;
6     c = c + 1;
7     p = 1;
8 p = c-k;
9 return 0;
```

Invalid: $EF(p = 0) \wedge EF(p = 1)$

Our Approach: “Dual Rewriting” with DRNLA



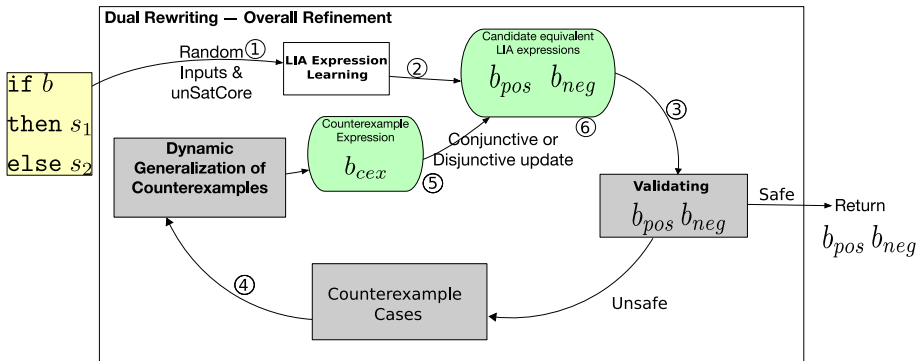
Our Approach: “Dual Rewriting” with DRNLA



Dual Rewriting algorithm

Challenge: Static tools struggle with NLAs.

- **Solution:** **dual re-writing** algorithm that mixes static and dynamic
- Iteratively synthesize LIA Boolean combinations denoted b_{pos} and b_{neg} .
- **Output:** equivalent LIA expressions of NLA expressions.



Challenges

- Identify all NLA expressions (b) from the program.
- How to know which cases to validate?
- How to do that within program context?

DRNLA Running Example

```
int y=1, z=6, c=0, p=2;
int k=*;
while (true):
  if (((z * z - 12 * y) - 6 * z) + 12) + c <= k):
    vtrace.if_35(k, n, y, z, c, p) ; //b_pos
  else:
    vtrace.else_35(k, n, y, z, c, p) ; //b_neg
    break;
  y = y + z;
  z = z + 6;
  c = c + 1;
  p = 1;
p = 0;
return 0;
```

//Concrete Traces

if_35; k; n; y; z; c; p

```
294; 78; 18487; 474; 78; 1
271; 57; 9919; 348; 57; 1
26; 8; 217; 54; 8; 1
296; 53; 8587; 324; 53; 1
...
```

else_35; k; n; y; z; c; p

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22; 23; 1657; 144; 23; 1
11; 12; 469; 78; 12; 1
21; 22; 1519; 138; 22; 1
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...
```

- b : original NLA expression.
- The goal is to an LIA expression b_{pos} , representing b as well as LIA expression b_{neg} representing $\neg b$.

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int y=1, z=6, c=0, p=2;
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- b : original NLA expression.
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- ① Initial guess, learned from random input (100).

$$b_{pos}(b) : \{2 \geq p, -p \leq -1, 0 = -6 \times c + z - 6, -p - z \leq -8, 0 \geq -c, 0 \geq c - k\}$$

$$b_{neg}(\neg b) : \{0 \geq -c + p, 0 \geq -k, 0 = -6 \times k + z - 12, 0 = p - 1, 0 = c - k - 1\}$$

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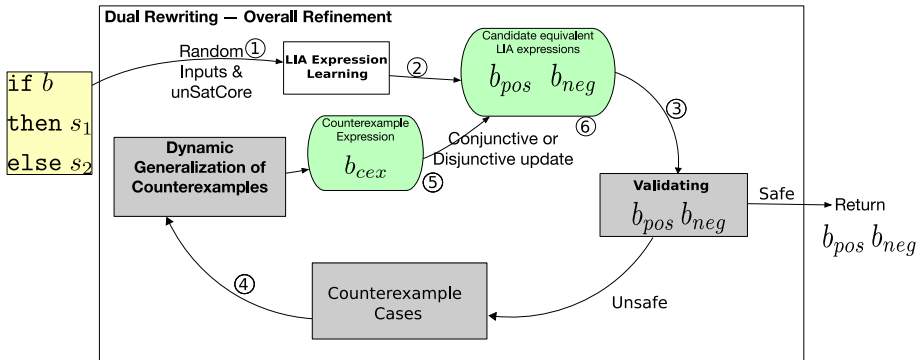
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2 Optimization process, remove identical ones, unsat core.

$$b_{pos} \equiv 0 \geq c - k \qquad b_{neg} \equiv 0 = c - k - 1$$



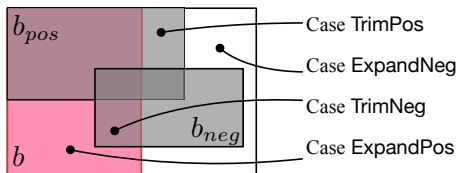
Transformation Example for Static Validation

```
1 int y=1, z=6, c=0, p=2;
2 int k=0;
3 while (true):
4
5
6
7
8
9
10
11
12
13
14 if (z*z-12y-6z+12+c > k):
15     break;
16 else:
17     y = y + z;
18     z = z + 6;
19     c = c + 1;
20     p = 1;
21 p = 0;
22 return 0;
```

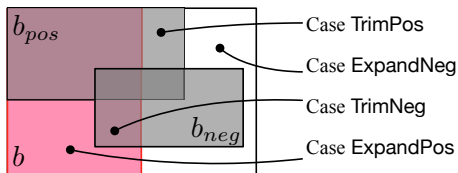
```
int y=1, z=6, c=0, p=2;
int k=0;
while(true):
    if (z*z - 12y - 6z + 12 + c > k && ¬b_pos):
        errorExpandPos // b_pos too small
    elif (z*z - 12y - 6z + 12 + c > k && b_neg):
        errorTrimNeg // b_neg too big
    elif (¬(z*z - 12y - 6z + 12 + c > k) && b_pos):
        errorTrimPos // b_pos too big
    elif (¬(z*z - 12y - 6z + 12 + c > k) && ¬b_neg):
        errorExpandNeg // b_neg too small
    if(z*z - 12y - 6z + 12 + c > k):
        break
    else:
        y = y + z;
        z = z + 6;
        c = c + 1;
        p = 1;
    p = 0;
    return 0;
```

Figure: Demonstration of instrumentation for static validation.

③ Static Validation for the Pair (b_{pos}, b_{neg}) .



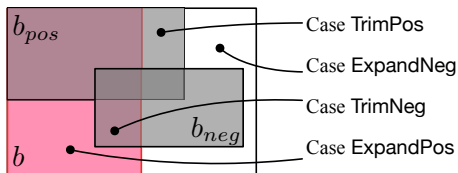
③ Static Validation for the Pair (b_{pos}, b_{neg}) .



b_{pos}

TrimPos, it includes executions where $\neg b$ holds and must be trimmed down.
ExpandPos, it does not include all executions where b does hold.

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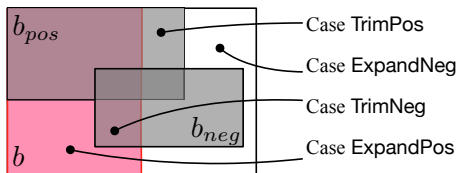
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TrimNeg, it includes executions where b holds and must be trimmed down.
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3 Static Validation for the Pair (b_{pos}, b_{neg}) .



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b_{neg}

TrimNeg, it includes executions where b holds and must be trimmed down.
ExpandNeg, it does not include all executions where $\neg b$ does hold.

$$b \equiv c \leq k, \quad b_{pos} \equiv 0 \geq c - k, \quad b_{neg} \equiv 0 = c - k - 1$$

For this running example, we need to ExpandNeg.

Counterexamples from Static Validation

```
ℓ 1: int y=1, z=6, c=0, p=2;
ℓ 2: int k=*;
ℓ 3: assume(true);
ℓ 12: assume(!(z*z-12y-6z+12+c>k));
ℓ 15: y = y + z;
ℓ 16: z = z + 6;
ℓ 17: c = c + 1;
ℓ 18: p = 1;
ℓ 3: assume(true);

ℓ 12: assume(!(z*z-12y-6z+12+c>k));
ℓ 15: y = y + z;
ℓ 16: z = z + 6;
ℓ 17: c = c + 1;
ℓ 18: p = 1;
ℓ 3: p = 1;
ℓ 8: assume(!(z*z-12y-6z+12+c>k));
ℓ 8: assume(bpos);
ℓ 9: errorExpandPos
```

Category: Reached `errorExpandPos`. So we need to expand. ($b_{pos} \rightarrow b_{neg}$)

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Counterexample: $c = 0, k = -2, p = 2, y = 1, z = 6$.

Another counterexample! $c = 0, k = -3, p = 2, y = 1, z = 6$.

Will there be more? Expand with generalization of this counterexample.

Dynamic Counterexample Generalization

Counterexample Models

y	z	c	k	p
1	6	0	-2	2
1	6	0	-3	2
1	6	0	-4	2
1	6	0	-5	2
1	6	0	-6	2
1	6	0	-7	2
1	6	0	-8	2

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1	6	0	-6	2
1	6	0	-7	2
1	6	0	-8	2

- ④ Output of DIG:
- We generate 1000 counterexample models using Z3.
 - Run DIG (dynamic learning) on all these counterexamples

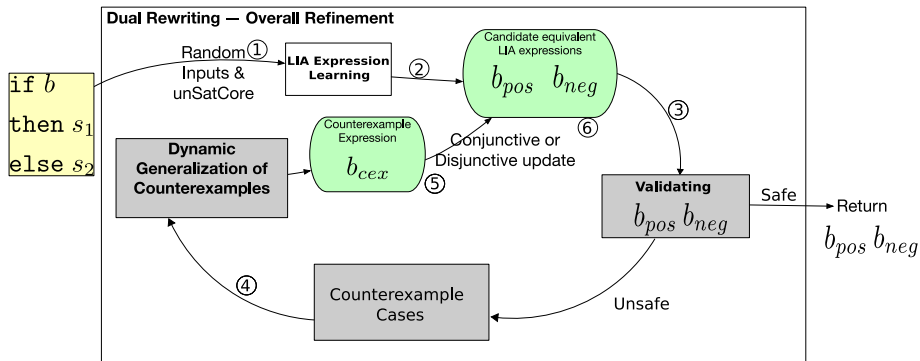
$$b_{cex} \equiv 0 \geq p + k \wedge 0 = p - 2 \wedge 0 = c \hat{\vee} b_{neg} \equiv 0 = c - k - 1$$

Dynamic Generalization Algorithm

```
1 procedure DYGENERALIZE( $b_{cur}$ ,  $ce_x$ , expand):  
2    $S := \text{getModels}(\text{formula}(ce_x), \text{iters}=1000);$   
3    $b_{ce_x} := \text{learn}(S);$   
4   if (expand):  
5     match UnsatCorePair( $b_{cur}, b_{ce_x}$ ) with  
6     | Some( $b_{usc}$ ) → return  $b_{usc}$   
7     | None → return  $b_{ce_x}$   
8   else:  
9     return  $b_{ce_x}$ 
```

- Encode counterexample with program context into formula.
- Generate more data with SMT solver Z3.
- Dynamic learn linear invariants from data.
- Return generalized counterexample with optimization.

We are back to the top of loop!



- ⑤ ExpandNeg, with the help of convex hull (b_{pos} and b_{neg} are updated).

$$b_{pos} \equiv 0 \geq c - k,$$

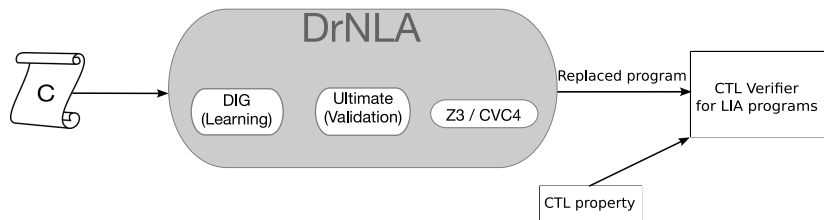
$$b_{neg} \equiv k - c \leq -1$$

- ⑥ Static validation returns correct, rewrite the program and prove CTL properties.

DrNLA Implementation

New Tool DrNLA

Built on top of DIG, Ultimate, Z3, CVC4, CIL, etc.



Nonlinear CTL Benchmarks

Note: No CTL Benchmarks for NLA programs, we built new ones!

CTLNLABench-DYNAMITE

NLA benchmarks in the Dynamite termination work ^a, insert CTL property $EF(p = 0) \wedge EF(p = 1)$ for all programs (56).

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Handcrafted Benchmarks

10 benchmarks with Combinations of NLA expressions with higher degrees, its LIA equivalent expressions of these NLA expressions are also more complex and involve disjunctions of linear constraints.

Learning Results

Example output of DRNLA on CTLNLABench-DYNAMITE

Benchmark	Source NLA	Output b_{pos}	Output b_{neg}
bresenham1-T.c	$\ell_{36} : 2Yx - 2X^2y + 2Y - v + c \leq k$	$0 \geq c - k$,	$k - c \leq -1$
cohencu2-T.c	$\ell_{32} : 3n^2 + 3n + 1 \leq k$	$0 \geq y - k$,	$k - y \leq -1$
egcd2-T.c	$\ell_{33} : c \geq xq + ys$	$0 \geq b - c$,	$-b + c \leq -1$

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Example output of DRNLA on CTLNLABench-PLDI13

Benchmark	Source NLA	Output b_{pos}	Output b_{neg}
afefp-T.c	$\ell_{19} : t^2 - 4s + 2t + 1 + c \leq k$	$0 \geq a - k$	$k - a \leq -1$
afegp-F.c	$\ell_{18} : t^2 - 4s + 2t + 1 + c \leq k$	$0 \geq a - k$	$k - a \leq -1$
afagp-T.c	$\ell_{21} : \neg(xz - x - y + 1 + c < k)$	$0 \geq -c + k$	$c - k \leq -1$

Rewrite Results

Table: DRNLA's rewrite results for handcrafted benchmarks

Benchmark	Res	T(s)	It.	Benchmark	Res	T(s)	It.
if-cubic-F.c	✓	51.5	3	square-loop-F.c	✓	383.6	12
if-cubic-T.c	✓	51.3	3	square-loop-T.c	✓	233.3	7
if-F.c	✓	78.8	6	while-cubic-F.c	✓	87.0	7
if-T.c	✓	76.5	6	while-cubic-T.c	✓	84.4	7
				while-F.c	✓	190.4	6
				while-T.c	✓	189.6	6

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				while-F.c	✓	190.4	6
				while-T.c	✓	189.6	6

An example result for 6 iterations (*if-F.c* | $p=2$).

ℓ_6 : $(36 == (x \times x)) \mapsto$

b_{pos} : $(((((0 + (x) <= 6) \wedge !(((0 == (p - 2)) \wedge ((-p) + x) <= -(1)))))) || ((x <= -(6)) \wedge (-p) <= -(2)) \wedge (8 >= (p - x)))) \wedge !(((0 == (p - 2)) \wedge (0 >= (p - x)) \wedge (3 >= (-p) + x))))$

b_{neg} : $(((((0 == (p - 2)) \wedge (0 >= -(x)) \wedge !(((2 >= p) \wedge (-x) <= -(6)) \wedge (4 >= (-p) + x)))))) || (((0 == (p - 2)) \wedge ((-p) + x) <= -(1)))) \wedge !(((x <= -(6)) \wedge (-p) <= -(2)) \wedge (8 >= (p - x)))) || (((0 == (p - 2)) \wedge (0 >= (p - x)) \wedge (3 >= (-p) + x))))$

Improvements on CTL Tools

Table: DRNLA's improvements for handcrafted benchmarks

Benchmark	Improve T2		Improve FUNCTION				Benchmark	Improve T2		Improve FUNCTION						
	Res	T(s)	DRNLA	FT	DRNLA	Res		T(s)	Res	T(s)	FT	DRNLA				
if-cubic-F.c	✓	0.9	X	0.7	??	0.1	??	0.1	X	0.8	✓	0.8	??	0.0	??	0.0
if-cubic-T.c	✓	0.6	✓	0.7	??	0.0	??	0.0	✓	0.8	✗	0.9	??	0.1	??	3.1
if-F.c	✓	0.7	X	0.7	??	0.0	??	0.1	✓	0.7	X	0.8	??	0.0	??	0.2
if-T.c	✓	0.7	✓	0.7	??	0.0	??	0.1	✓	0.7	✓	0.7	??	0.0	??	0.5
									X	0.9	X	0.9	??	0.1	??	0.4
									✓	0.7	✓	0.8	??	0.1	??	3.2

Definition (Improvements)

Improvements meaning number of benchmarks can be proved now to the total benchmarks ratio.

Experiment Summary

Before DRNLA

- Existing tools simply do not support these NLA programs
- FUNCTION returns Unknown
- T2 is Unsound

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- T2 is Unsound

With DRNLA

- CTLNLABench-DYNAMITE:
 - T2 can now analyze **26 out of the 56** new benchmarks
 - FUNCTION correctly analyzes **12 out of the 56** new benchmarks
- CTLNLABench-PLDI13:
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- Handcrafted Benchmarks:
 - T2 can now analyze **6 out of the 10** new benchmarks

Experiment Summary

Before DRNLA

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 - T2 can now analyze **6 out of the 10** new benchmarks
- DRNLA's rewrite does not decrease quality of the CTL tools.
- These improvements come with almost no additional runtime cost.

Conclusions

Contributions and Findings

- CTL properties can indeed be verified with NLA programs.
- Dual rewriting technique effectively synthesizes boolean combinations of linear expressions that are equivalent to its NLA counterpart.
- DRNLA can be used as a pre-processing step before CTL verification.
- Static verification tools were often useful at validating whether NLA expressions are equivalent to provided LIA alternatives.

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Future work

- Desired shape of NLA expressions.
- Convergence of DRNLA.
- Other reasons that hinder CTL verification tools.
- Applications in code efficiency, compiler optimization.

Q & A
Thank You!