Properties of sets

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Properties	Set notation
Commutative	$A \cup B = B \cup A$; $A \cap B = B \cap A$.
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$. (same with $\cap$)
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Complement	$A \cup \overline{A} = U$.
Idempotent	$A \cap A = A$; $A \cup A = A$.

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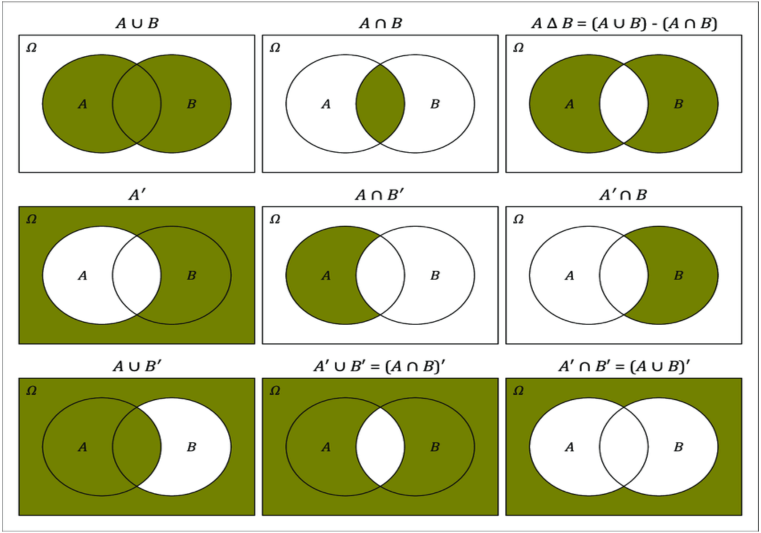
Laws

- De Morgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Law of identity: $A \cup \emptyset = A$; $A \cap \emptyset = A$. ($\emptyset$ is the identity of U)

Venn diagram



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[1] https://doi.org/10.5334/jcaa.16

Proof by contradiction

Proof by contradiction in proposition logic

1. Assume $\neg P$.
2. Show $\neg P \to \bot$.
3. Conclude contradiction, refute $\neg P$.

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2. Show $\neg P \rightarrow \bot$.
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Proof by contradiction in sets

1. Assume $x \in S$.
2. Show $y \in U \wedge y \notin U$.
3. Conclude assumption is false $\rightarrow x \notin S$.

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x . x \in \overline{A} \cap \overline{B}$. [assumption]

□

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x. x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]

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Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x . x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \land x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]

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Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x . x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]
4. $x \in A \cup B \rightarrow x \in A \vee x \in B$. [defn complement, ∪] [Proof by contradiction second step]

□

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x. x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]
4. $x \in A \cup B \rightarrow x \in A \vee x \in B$. [defn complement, $\cup$] [Proof by contradiction second step]
   - $x \in A$, contradicts with step 2 ($x \notin A$).

□

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x. x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]
4. $x \in A \cup B \rightarrow x \in A \vee x \in B$. [defn complement, $\cup$] [Proof by contradiction second step]
    - $x \in A$, contradicts with step 2 ($x \notin A$).
    - $x \in B$, contradicts with step 2 ($x \notin B$).

□

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x. x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]
4. $x \in A \cup B \rightarrow x \in A \vee x \in B$. [defn complement, $\cup$]
   [Proof by contradiction second step]
   - $x \in A$, contradicts with step 2 ($x \notin A$).
   - $x \in B$, contradicts with step 2 ($x \notin B$).
5. $x \notin \overline{A \cup B}$ is false $\rightarrow x \in \overline{A \cup B}$. [Proof by contradiction third step]

□

Exercise

Claim: Right side-to-left direction of DeMorgan's Law, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Proof.

1. $\forall x. x \in \overline{A} \cap \overline{B}$. [assumption]
2. $x \notin A \wedge x \notin B$. [defn complement]
3. assume $x \notin \overline{A \cup B}$. [Proof by contradiction first step]
4. $x \in A \cup B \rightarrow x \in A \vee x \in B$. [defn complement, $\cup$]
   [Proof by contradiction second step]
   - $x \in A$, contradicts with step 2 ($x \notin A$).
   - $x \in B$, contradicts with step 2 ($x \notin B$).
5. $x \notin \overline{A \cup B}$ is false $\rightarrow x \in \overline{A \cup B}$. [Proof by contradiction third step]
6. $\forall x. x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A \cup B}$, by subset definition it equals to $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, therefore the claim is valid.

□

Q & A